Name: ________________________ Date: __________________

Learning Goal 3.3	Using more derivative rules.

More Questions – Solutions

1. Find the third derivative of the following functions.

a. $f(x) = \dfrac{1}{x^5}$

$$f(x) = x^{-5}$$

$$f'(x) = -5x^{-6}$$

$$f''(x) = -5(-6x^{-7}) = 30x^{-7}$$

$$f'''(x) = 30(-7x^{-8}) = -\frac{210}{x^8}$$

b. $g(x) = -4x^5 + 3x^2 - 5/x^2$

$$g(x) = -4x^5 + 3x^2 - 5x^{-2}$$

$$g'(x) = -20x^4 + 6x + 10x^{-3}$$

$$g''(x) = -80x^3 + 6 - 30x^{-4}$$

$$g'''(x) = -240x^2 + 120x^{-5} = -120\left(2x^2 + \frac{1}{x^5}\right) = -120\left(\frac{2x^7 + 1}{x^5}\right)$$

c. $h(x) = (4 - x)^3$

$$h'(x) = -3(4 - x)^2$$

$$h''(x) = 6(4 - x) = 24 - 6x$$

$$h'''(x) = -6$$

d. $f(x) = \sin(x^2 + 1)$

$$f'(x) = \cos(x^2 + 1) \times 2x = 2x\cos(x^2 + 1)$$

$$f''(x) = 2\cos(x^2 + 1) + 2x(-\sin(x^2 + 1) \times 2x) = 2\cos(x^2 + 1) - 4x^2\sin(x^2 + 1)$$

$$f'''(x) = -2\sin(x^2 + 1) + (8x\sin(x^2 + 1) + 4x^2 \times \cos(x^2 + 1) \times 2x) = -2\sin(x^2 + 1) + 8x\sin(x^2 + 1) + 8x^3\cos(x^2 + 1)$$

e. $y = \frac{1}{(1-4x^3)^2}$

$$y = (1-4x^3)^{-2}$$

$$\frac{dy}{dx} = -2(1-4x^3)^{-3} \times -12x^2$$
$$= 24x^2(1-4x^3)^{-3}$$

$$\frac{d^2y}{dx^2} = 48x(1-4x^3)^{-3} + 24x^2 \times -3(1-4x^3)^{-4} \times -12x^2$$
$$= 48x(1-4x^3)^{-3} + 864x^4(1-4x^3)^{-4}$$
$$= 48x(1-4x^3)^{-4}(1-4x^3+18x^3)$$
$$= 48x(1-4x^3)^{-4}(1+14x^3)$$

$$\frac{d^3y}{dx^3} = 48(1-4x^3)^{-4}(1+14x^3) + 48x(1+14x^3) \times (-4(1-4x^3)^{-5} \times -12x^2)$$
$$+ 48x(1-4x^3)^{-4} \times 42x^2$$
$$= 48(1-4x^3)^{-4}(1+14x^3) + 2\,304x^3(1+14x^3)(1-4x^3)^{-5} + 2\,016x^3(1-4x^3)^{-4}$$
$$= 48(1-4x^3)^{-5}\big((1-4x^3)(1+14x^3) + 48x^3(1-14x^3) + 42x^3(1-4x^3)\big)$$
$$= 48(1-4x^3)^{-5}\big((1+10x^3-56x^6) + (48x^3-672x^6) + (42x^3-168x^6)\big)$$
$$= 48(1-4x^3)^{-5}((1+100x^3-896x^6))$$

f. $g(x) = x\cos\left(\frac{1}{x}\right)$

$$g'(x) = \cos\left(\frac{1}{x}\right) + \left(x \times -\sin\left(\frac{1}{x}\right) \times -x^{-2}\right)$$
$$= \cos\left(\frac{1}{x}\right) + \frac{1}{x}\sin\left(\frac{1}{x}\right)$$

$$g''(x) = \left(-\sin\left(\frac{1}{x}\right) \times -x^{-2}\right) + \left(\frac{1}{x}\cos\left(\frac{1}{x}\right) \times -x^{-2} + \sin\left(\frac{1}{x}\right) \times -x^{-2}\right)$$
$$= \frac{1}{x^2}\sin\left(\frac{1}{x}\right) - \frac{1}{x^3}\cos\left(\frac{1}{x}\right) - \frac{1}{x^2}\sin\left(\frac{1}{x}\right)$$

$$g'''(x) = \left(-2x^{-3}\sin\left(\frac{1}{x}\right) + \frac{1}{x^2}\cos\left(\frac{1}{x}\right) \times -x^{-2}\right) - \left(-3x^{-4}\cos\left(\frac{1}{x}\right) + \frac{1}{x^3} \times -\sin\left(\frac{1}{x}\right) \times -x^{-2}\right)$$
$$- \left(-2x^{-3}\sin\left(\frac{1}{x}\right) + \frac{1}{x^2}\cos\left(\frac{1}{x}\right) \times -x^{-2}\right)$$
$$= \left(-\frac{2}{x^3}\sin\left(\frac{1}{x}\right) - \frac{1}{x^4}\cos\left(\frac{1}{x}\right)\right) - \left(-\frac{3}{x^4}\cos\left(\frac{1}{x}\right) + \frac{1}{x^5}\sin\left(\frac{1}{x}\right)\right) - \left(-\frac{2}{x^3}\sin\left(\frac{1}{x}\right) - \frac{1}{x^4}\cos\left(\frac{1}{x}\right)\right)$$
$$= -\frac{2}{x^3}\sin\left(\frac{1}{x}\right) - \frac{1}{x^4}\cos\left(\frac{1}{x}\right) + \frac{3}{x^4}\cos\left(\frac{1}{x}\right) - \frac{1}{x^5}\sin\left(\frac{1}{x}\right) + \frac{2}{x^3}\sin\left(\frac{1}{x}\right) + \frac{1}{x^4}\cos\left(\frac{1}{x}\right)$$
$$= \sin\left(\frac{1}{x}\right)\left(-\frac{2}{x^3} - \frac{1}{x^5} + \frac{2}{x^3}\right) + \cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^4} + \frac{3}{x^4} + \frac{1}{x^4}\right)$$
$$= \frac{3}{x^4}\cos\left(\frac{1}{x}\right) - \frac{1}{x^5}\sin\left(\frac{1}{x}\right)$$

g. $y^2 = 1 + x^2$

$$2y\frac{dy}{dx} = 2x$$
$$\frac{dy}{dx} = \frac{2x}{2y}$$
$$= \frac{x}{y}$$

$$\frac{d^2y}{dx^2} = \frac{(2y \times 2) - \left(2x \times 2\frac{dy}{dx}\right)}{4y^2}$$
$$= \frac{4y - 4x\left(\frac{x}{y}\right)}{4y^2}$$
$$= \frac{4y - \frac{4x^2}{y}}{4y^2}$$
$$= \frac{4y^2 - 4x^2}{4y^3}$$

$$\frac{d^3y}{dx^3} = \frac{4y^3\left(8y\frac{dy}{dx} - 8x\right) - (4y^2 - 4x^2)\left(12y^2\frac{dy}{dx}\right)}{16y^6}$$
$$= \frac{\left(32y^4\frac{dy}{dx} - 32xy^3\right) - (4y^2 - 4x^2)\left(12y^2\frac{dy}{dx}\right)}{16y^6}$$
$$= \frac{(32y^4 - 48y^4 + 48x^2y^2)\frac{dy}{dx} - 32xy^3}{16y^6}$$
$$= \frac{(32y^4 - 48y^4 + 48x^2y^2)\left(\frac{x}{y}\right) - 32xy^3}{16y^6}$$
$$= \frac{16xy(2y^2 - 3y^2 + 3x^2 - 3y^2)}{16y^6}$$
$$= \frac{x(2y^2 - 3y^2 + 3x^2 - 3y^2)}{y^5}$$

h. $\sqrt{x}+\sqrt{y}=9$

$$x^{1/2}+y^{1/2}=9$$

$$\frac{1}{2}x^{-3/2}+\frac{1}{2}y^{-3/2}\frac{dy}{dx}=0$$
$$\frac{1}{2}y^{-3/2}\frac{dy}{dx}=-\frac{1}{2}x^{-3/2}$$
$$\frac{dy}{dx}=\frac{-\frac{1}{2}x^{-3/2}}{\frac{1}{2}y^{-3/2}}$$
$$\frac{dy}{dx}=-x^{-3/2}y^{3/2}$$

$$\frac{d^2y}{dx^2}=\frac{3}{2}x^{-5/2}y^{3/2}-x^{-3/2}\times\frac{3}{2}y^{1/2}\frac{dy}{dx}$$
$$\frac{d^2y}{dx^2}=\frac{3}{2}x^{-5/2}y^{3/2}-\frac{3}{2}x^{-3/2}y^{1/2}\left(-x^{-3/2}y^{3/2}\right)$$
$$\frac{d^2y}{dx^2}=\frac{3}{2}x^{-5/2}y^{3/2}+\frac{3}{2}x^{-3}y^{2}$$

$$\frac{d^3y}{dx^3}=\frac{3}{2}\left(-\frac{5}{2}x^{-7/2}y^{3/2}+\frac{3}{2}y^{1/2}\frac{dy}{dx}x^{-5/2}\right)+\frac{3}{2}\left(-3x^{-4}y^2+2yx^{-3}\frac{dy}{dx}\right)$$
$$=-\frac{15}{4}x^{-7/2}y^{3/2}+\frac{9}{4}y^{1/2}\frac{dy}{dx}x^{-5/2}-\frac{9}{2}x^{-4}y^2+3yx^{-3}\frac{dy}{dx}$$
$$=-\frac{15}{4}x^{-7/2}y^{3/2}-\frac{9}{2}x^{-4}y^2+\left(\frac{9}{4}y^{1/2}x^{-5/2}+3yx^{-3}\right)\frac{dy}{dx}$$
$$=-\frac{15}{4}x^{-7/2}y^{3/2}-\frac{9}{2}x^{-4}y^2+\left(\frac{9}{4}y^{1/2}x^{-5/2}+3yx^{-3}\right)\left(-x^{-3/2}y^{3/2}\right)$$
$$=-\frac{15}{4}x^{-7/2}y^{3/2}-\frac{9}{2}x^{-4}y^2-\left(\frac{9}{4}y^{2}x^{-4}+3y^{5/2}x^{-9/2}\right)$$
$$=-\frac{15y^{3/2}}{4x^{7/2}}-\frac{9y^2}{2x^4}-\frac{9y^2}{4x^4}-\frac{3y^{5/2}}{x^{9/2}}$$
$$=-\frac{3y^{3/2}}{x^{9/2}}\left(\frac{5x}{4}+\frac{3(xy)^{1/2}}{2}+\frac{3(xy)^{1/2}}{4}+y\right)$$
$$=-\frac{3y^{3/2}}{x^{9/2}}\left(\frac{5x+6(xy)^{1/2}+3(xy)^{1/2}+4y}{4}\right)$$
$$=-\frac{3\sqrt{xy^3}}{4x^5}\left(5x+9\sqrt{xy}+4y\right)$$