Calculus 12 | Word Problems | Limits and Derivatives

Name: ________________________ Date: __________________

Learning Goal 2.3	Creating confidence in word problems.

Example The distance travelled by a free – falling object can be calculated by using the formula $s(t) = 4.9t^2$, where s represents the distance travelled in metres after t seconds. If a rock is dropped from the top of a 500 – metre cliff,

a. Find the average velocity from:
   i. 4 seconds to 4.1 seconds
   ii. 4 seconds to 4.01 seconds
   iii. 4 seconds to 4.001 seconds

b. Estimate the instantaneous velocity at 4 seconds.

39.2 m/s

i. $s(4) = 4.9(4)^2 = 78.4$ $s(4.1) = 4.9(4.1)^2 = 82.369$

$m = \frac{82.369 - 78.4}{4.1 - 4} = 39.69$ m/s

ii. $s(4) = 78.4$ $s(4.01) = 4.9(4.01)^2 = 78.79$

$m = \frac{78.79 - 78.4}{4.01 - 4} = 39.25$ m/s

iii. $s(4) = 78.4$ $s(4.001) = 4.9(4.001)^2 = 78.44$

$m = \frac{78.44 - 78.4}{4.001 - 4} = 39.20$ m/s

Example A manufacturer produces bolts of fabric with a fixed width. The cost of producing x yards of this fabric is $C = f(x)$ dollars.

a. What is the meaning of the derivative, $f'(x)$? What are its units?
b. In practical terms, what does it mean to say that $f'(1000) = 9$?
c. Which is greater, $f'(50)$ or $f'(500)$?

a. $f'(x)$ can also be written as $\frac{dC}{dx}$ and then the units become more obvious ... I think

$\frac{dC}{dx}$ = cost to produce x yards

cost in \$

yards produced

b. $f'(1000) = 9 \Rightarrow$ If we create 1000 yds of fabric, it will cost \$9/yard to produce.

c. Since the cost of production generally decreases with the amount of product produced,

$f'(50) < f'(500)$

Example An object moves in a straight line with its position at time t seconds given by $s(t) = -t^2 + 8t$, where s is measured in metres. Find the velocity when $t = 0, t = 4$ and $t = 6$.

$$s'(t) = \lim_{h\to 0} \frac{(-(t+h)^2 + 8(t+h)) - (-t^2+8t)}{h}$$

$$= \lim_{h\to 0} \frac{-(t^2+2th+h^2) + 8t + 8h + t^2 - 8t}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{-t^2} - 2th - h^2 \cancel{+8t} + 8h \cancel{+t^2} \cancel{-8t}}{h}$$

$$= \lim_{h\to 0} \frac{-2th - h^2 + 8h}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(-2t-h+8)}{\cancel{h}}$$

$$= \lim_{h\to 0} -2t - h + 8$$

$$s'(t) = -2t + 8$$

$$s'(t) = -2t+8$$

$$s'(0) = -2(0)+8 = 8 \text{ m/s}$$

$$s'(4) = -2(4)+8 = -8+8 = 0 \text{ m/s}$$

$$s'(6) = -2(6)+8 = -12+8 = -4 \text{ m/s}$$

Example Find an equation of the line that is tangent to the graph of $f(x) = \sqrt{x+1}$ and parallel to $x - 6y + 4 = 0$.

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \times \frac{\sqrt{x+h+1}+\sqrt{x+1}}{\sqrt{x+h+1}+\sqrt{x+1}}$$

$= \frac{0}{0}$ indeterminate form so we need to do some algebra

$$= \lim_{h\to 0} \frac{(x+h+1) - (x+1)}{h(\sqrt{x+h+1}+\sqrt{x+1})}$$

$$= \lim_{h\to 0} \frac{\cancel{x}+h\cancel{+1}\cancel{-x}\cancel{-1}}{h(\sqrt{x+h+1}+\sqrt{x+1})}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h+1}+\sqrt{x+1})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h+1}+\sqrt{x+1}}$$

$$= \frac{1}{\sqrt{x+1}+\sqrt{x+1}}$$

$$= \frac{1}{2\sqrt{x+1}}$$

$$x - 6y + 4 = 0$$
$$6y = x + 4$$
$$y = \frac{1}{6}x + \frac{2}{3}$$

↑ so the slope we're trying to match is $\frac{1}{6}$

$$\frac{1}{2\sqrt{x+1}} = \frac{1}{6}$$

$$\frac{2\sqrt{x+1}}{2} = \frac{6}{2}$$

$$(\sqrt{x+1})^2 = (3)^2$$

$$x + 1 = 9$$
$$-1 \quad -1$$
$$x = 8$$

$$y = \sqrt{8+1} = \sqrt{9} = 3$$

$$y - 3 = \frac{1}{6}(x-8)$$

$$y - 3 = \frac{1}{6}x - \frac{4}{3}$$

$$\boxed{y = \frac{1}{6}x + \frac{5}{3}}$$

Example A football is kicked up into the air. Its height, h, above the ground in metres at t seconds can be modelled by $h(t) = 18t - 4.9t^2$. Determine $h'(2)$. What does this represent?

$$h'(2) = \lim_{t\to 2} \frac{h(t) - h(2)}{t-2}$$
$$= \lim_{t\to 2} \frac{18t - 4.9t^2 - 16.4}{t-2}$$
$$= \lim_{t\to 2} \frac{\cancel{(t-2)}(-4.9t+8.2)}{\cancel{t-2}}$$
$$= \lim_{t\to 2} -4.9t + 8.2$$
$$= -4.9(2) + 8.2$$
$$= -1.6 \text{ m/s}$$

$h(2) = 18(2) - 4.9(2)^2$
$= 36 - 19.6$
$= 16.4$

2	-4.9	18	-16.4
	$\downarrow$	-9.8	16.4
	-4.9	8.2	0

The velocity of the football after 2 seconds is -1.6 m/s.

Example At what point on the graph of $y = x^2 - 4x - 5$ is the tangent parallel to $2x - y = 1$?

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{((x+h)^2 - 4(x+h) - 5) - (x^2 - 4x - 5)}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{4x} - 4h - \cancel{5} - \cancel{x^2} + \cancel{4x} + \cancel{5}}{h}$$
$$= \lim_{h\to 0} \frac{2xh + h^2 - 4h}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h-4)}{\cancel{h}}$$
$$= \lim_{h\to 0} 2x + h - 4$$
$$= 2x - 4$$

$2x - y = 1$
$y = 2x - 1$

the slope we're aiming for

$2x - 4 = 2$
$+4 \quad +4$
$\frac{2x}{2} = \frac{6}{2}$
$x = 3$

$y = (3)^2 - 4(3) - 5$
$= 9 - 12 - 5$
$= -8$

$y + 8 = 2(x-3)$
$y + 8 = 2x - 6$
$\boxed{y = 2x - 14}$

Example Determine the equations of both lines that are tangent to the graph of $f(x) = x^2$ and pass through the point $(1, -3)$.

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$
$$= \lim_{h\to 0} \frac{2xh + h^2}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

Slope btwn (x, x^2) and $(1, -3)$ will be the same as the derivative.

$\cancel{(x-1)} \times \frac{x^2+3}{\cancel{x-1}} = 2x \times (x-1)$

$x^2 + 3 = 2x^2 - 2x$
$-x^2 - 3 \quad -x^2 \quad -3$
$0 = x^2 - 2x - 3$
$0 = (x-3)(x+1)$

$$= \lim_{h \to 0} 2x + h$$
$$= 2x$$

$x - 3 = 0$
$x = 3$

$y = (3)^2$
$= 9$

Slope $= 2(3)$
$= 6$

$y - 9 = 6(x-3)$
$y - 9 = 6x - 18$
$\boxed{y = 6x - 9}$

$x + 1 = 0$
$x = -1$

$y = (-1)^2$
$= 1$

Slope $= 2(-1)$
$= -2$

$y - 1 = -2(x+1)$
$y - 1 = -2x - 2$
$\boxed{y = -2x - 1}$

Example For the function $f(x) = x|x|$, show that $f'(0)$ exists. What is the value?

$$= \begin{cases} -x^2 & x < 0 \\ x^2 & x > 0 \end{cases}$$

the derivative from the left

$$f'(0) = \lim_{x \to 0^-} \frac{(-x^2) - (-(0)^2)}{x - 0}$$
$$= \lim_{x \to 0^-} \frac{-x^2}{x}$$
$$= \lim_{x \to 0^-} -x$$
$$= 0$$

the derivative from the right.

$$f'(0) = \lim_{x \to 0^+} \frac{(x^2) - (0^2)}{x - 0}$$
$$= \lim_{x \to 0^+} \frac{x^2}{x}$$
$$= \lim_{x \to 0^+} x$$
$$= 0$$

they match. So the derivative at $x = 0$ not only exists,

$f'(0) = 0$.